Evidence of discrete scale invariance in DLA and time-to-failure by canonical averaging

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Abstract

Discrete scale invariance, which corresponds to a partial breaking of the scaling symmetry, is reflected in the existence of a hierarchy of characteristic scales $l_0, l_0\lambda, l_0\lambda^2, ...$ where λ is a preferred scaling ratio and l_0 a microscopic cut-off. Signatures of discrete scale invariance have recently been found in a variety of systems ranging from rupture, earthquakes, Laplacian growth phenomena, "animals" in percolation to financial market crashes. We believe it to be a quite general, albeit subtle phenomenon. Indeed, the practical problem in uncovering an underlying discrete scale invariance is that standard ensemble averaging procedures destroy it as if it was pure noise. This is due to the fact, that while λ only depends on the underlying physics, l_0 on the contrary is realisation-dependent. Here, we adapt and implement a novel so-called "canonical" averaging scheme which re-sets the l_0 of different realizations to approximately the same value. The method is based on the determination of a realization-dependent effective critical point obtained from, *e.g.*, a maximum susceptibility criterion. We demonstrate the method on diffusion limited aggregation and a model of rupture.

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1 Introduction

With only a finite number N of independent random variables, the first moment of the distribution can only be determined with a precision which is of the order of $1/\sqrt{N}$. More generally, when introducing interactions between random variables, small systems will be dominated by fluctuations while it is often the case that infinite systems have well-defined observables and are so-called "self-averaging". As a consequence, phase transitions and critical points are only well-defined in the infinite size limit. For instance, the critical value $p_c = 0.249$ for simple-cubic bond percolation is only true for an infinite system [1]. However, both numerical simulations and experiments deal with finite realisations or systems. This means that various quantities of the finite system will obviously be realization dependent. Thus percolation in finite systems is not simply characterized by a single number p_c but instead by a distribution of p_c's belonging to an ensemble of finite realizations generated by the same statistics [2]. The situation may be even more delicate in the presence of some frozen disorder where the infinite size limit may not be defined uniquely, hence needing a description in terms of a distribution ("no-self-averaging") as in spin glasses [3].

The standard strategy to deal with finite size effects is to average over a (hopefully large) set of system realizations. In percolation, this means that for a fixed system size and fixed probability p of bond occupancy, one generates as many realizations possible and then averages the relevant quantities over this ensemble of different realizations. However, it has recently been pointed out that this "grand-canonical" [1] ensemble averaging may hide important physics. This will be the case especially in disordered systems, due to the introduction of spurious noise of relative amplitude proportional to the inverse square root of the system volume [4]. The "grand canonical" averaging may thus destroy the information embedded in the fluctuations in finite systems, controlled by correlation lengths with exponent less than $\frac{2}{d}$ ($\frac{2}{d}$ is the minimum value of the correlation length exponent that would not be hidden by the usual grand canonical averaging [4]).

In fact, there is a well-known example which illustrates dramatically how averaging can destroy the most important physical information [5] (see also [6] for a pedagogical introduction). Consider a wave propagating within a heterogeneous medium. The wave amplitude obeys the wave hyperbolic equation with quenched random coefficients. The natural quantity characterizing the propagation is the Green function of this equation. It turns out that its average over statistically equivalent sample realizations decays exponentially with the distance from the source with a characteristic decay length l_e equal to the mean free path. Thus, at distances from the source larger than l_e, the ensemble averaged Green function is completely smeared out simply because of the destructive interference between the random phases of different system realizations, *i.e*, the phases are only coherent over a distance l_e and become

[1]This term refers to the fact that the number of bonds is not the same across the different realizations. This is thus similar to the grand-canonical ensemble in statistical physics where p in our percolation example plays the role of an effective chemical potential for the bond occupancy.

incoherent at larger distances. As a consequence, the ensemble averaging destroys all information on the phases in the specific realizations and in the same token destroys the information on the transport properties at distances larger than l_e. Beyond l_e, the wave does not disappear as would be concluded from the examination of the ensemble averaged Green function, but enters a new diffusive regime characterized by the *non-averaged* Green function, as shown by Anderson [7]. Realizing this, it is now possible to define a quantity such that its average describes correctly the diffusive regime beyond l_e. This quantity is the so-called double Green function defined as the product of the advanced and the retarded Green functions, which is defined in order to "synchronize" the configuration dependent phases.

Another example of recent interest is found in the new concept of DNA walks [8, 9] which shows the relation between the local content of nucleotide pairs composition along a DNA molecule. A recent work [10] demonstrates that the existence of DNA phases in the codons, each of which consists of three base-pairs, requires that the statistical analysis of DNA walks should be done in the proper DNA phases, respecting positions in codons. Otherwise, coding trends of DNA fragments could compensate each other. This example can be persued using an analogy with language (D. Stauffer, private communication). Imagine one wants to understand the use of an alphabet by looking at the words in a dictionary. One will not find much relevant information by putting all the words together and look for correlations of the letters. However if ones looks at the words themselves, taking into account the position of the letters in each word, then one will find important regularities much more easily.

In order to address a somewhat similar problem in critical phenomena, Pazmandi *et al.* have recently proposed an alternative averaging procedure to the usual ensemble, or grand-canonical, averaging. They have coined it "canonical averaging" [4], since it consists of identifying, for *each* system realization, the specific value of the critical control parameter p_c^R. The natural parameter to study then becomes

$$\Delta = \left| p - p_c^R \right| \tag{1}$$

instead of $|p - \langle p_c \rangle|$. p denotes the control parameter of the system and the averaging is to be over the different realisations for the same values of Δ. This procedure has been tested successfully on the Mott-Insulator to Superfluid transition of interacting bosons with infinite hopping range in a random potential at zero temperature for which exact renormalization group equations are available [4]. In quenched random Ising and Ashkin-Teller models, Wiseman and Domany have recently shown in addition that sample-to-sample fluctuations of the susceptibility maximum measured in finite systems are smaller by a factor of 70 than those of the susceptibility measured in the same finite systems at the special value of the temperature corresponding to the critical point of an infinite system [11]. This led them to suggest a more efficient simulation method to estimate exponents from finite size scaling analysis: instead of the accepted procedure of performing simulations for all sizes at one value p_c, perform simulations at various values of p around p_c in order to identify the sample-

dependent maximum of the susceptibility, *i.e.*, the sample-dependent critical value p_c^R. They then use the sample-dependent reduced control $p - p_c^R$ to carry out the averaging. Similarly, Ballesteros et al. [12] studied the site-diluted Ising model in three dimensions and made the important point that only quantities measurable on the same finite lattice must appear in the finite size scaling Ansatz of observables.

The present paper is concerned with the implementation of these ideas for the detection of complex exponents and its associated log-periodicity in critical systems.

The understanding of complex exponents has advanced significantly [13, 14] in the last few years. Complex exponents reflect a discrete scale invariance (DSI), *i.e.*, the fact that symmetry with respect to dilation only occurs under magnification under special factors, which are arbitrary (integer) powers λ^n of a preferred scaling ratio λ. Complex exponents have been studied in the eighties in relation to various problems of physics embedded in hierarchical systems. Only recently, has it been realized that discrete scale invariance and its associated complex exponents may appear "spontaneously" in euclidean systems and without the need for a pre-existing hierarchy. Systems that have been found to exhibit self-organized DSI are Laplacian growth models [15, 16], rupture in heterogeneous systems [17, 18, 19], animals [20] (a generalization of percolation), possibly in earthquakes [21] among other systems [13]. In addition, general field theoretical arguments [20] indicate that complex exponents are to be expected generically for out-of-equilibrium and quenched disordered systems. Log-periodic structures indicate that the system and its underlying physical mechanisms have characteristic length scale(s). This is extremely interesting since it may provide important information on the physical mechanism and hence can be used as a important modelling constraint. Indeed, simple power law behaviors can apparently be found everywhere, as seen from the explosion of the concepts of fractals, criticality and self-organized-criticality. As an example, the power law distribution of earthquake energies, known as the Gutenberg-Richter law, can be obtained by many different mechanisms and from a variety of models. Hence, its usefulness as a modelling constraint is rather doubtful. A power law reflects the absence of characteristic scales. This is a desirable property in the quest for universal behaviors but useless for retrieving the underlying physical mechanisms and even more so the sample specific signatures useful for instance for prediction. In contrast, the presence of log-periodic features may teach us about important physical structures, which would otherwise be hidden in a description, which is (falsly) fully scale invariant

A problem however, when identifying log-periodic signatures in some observable due to an underlying discrete scale invariance, is that the phase of the oscillations "moves" as the length of the signal over which the averaging is performed is changed. They thus have an aspect of noise. However, previous numerical simulations on Laplacian growth models [15] and renormalization group calculations [20] have taught us that the presence of real noise modifies the phase in the log-periodic oscillations in a sample specific way leading to a "destructive interference" between different realisations. This is similar to the scrambling of the random phases between different system realizations of a wave propagating in a random system mentioned above.

Hence, a first strategy is to carry out an analysis on each sample realization separately and *then* average over the different realisations, as done in our previous work on DLA [15]. An enticing alternative is to introduce a new averaging scheme that does not destroy the oscillations, based on the preceding considerations [4]. Our purpose here is thus to adapt the averaging scheme proposed in [4] in terms of a realization-dependent effective critical point obtained from, *e.g.*, a maximum susceptibilty criterion, to the detection of log-periodicity in Laplacian growth processes and in rupture. The hope is that the use of a realization-dependent effective critical point should lead to a "rephasing" of the log-oscillations in the averaging process.

We first examine the case of diffusion-limited-aggregation clusters and the calculation of their complex fractal dimensions. We then turn to the implementation of the canonical averaging in the time-to-failure analysis of a dynamical model of rupture.

2 Laplacian growth models

Diffusion-limited-aggregation (DLA) is a growth model introduced by Witten and Sander [22] in which particles are introduced one after another from large distances and from there perform a random walk until they either reach the perimeter of the growing cluster and stick to it or are removed if they get too far away from the center. DLA has been much studied as a paradigm of the spontaneous formation of complex fractal patterns [23]. Here, the fractal (capacity) dimension is defined by

$$M(r) \propto r^D, \tag{2}$$

where $M(r)$ is the "mass" of the cluster or, similarly, the number of particles within a disk of radius r centered on the initial cluster seed. The numerical value of the fractal dimension of off-lattice DLA-clusters have been reported to be $D \approx 1.71 \pm 0.02$ [24, 25] and $D \approx 1.60 \pm 0.02$ [26] depending on how D is estimated [26]. A reason for the surprisingly large deviation in the dimension reported by different authors may be the presence of log-periodic oscillations in the local fractal dimension D_r as a function of r, or equivalently M, as shown in [15]. Log-periodic oscillations will make estimates of D rather sensitive to how they are calculated and especially on how the averaging over the different realisations is performed. A similar phenomenon [2] has been pointed out for the determination of the effective diffusion exponent versus time of random walks with a fixed bias direction on randomly diluted cubic lattices above the percolation threshold [27]. The observed log-periodicity make it difficult to determine the diffusion exponent for large bias.

Despite their apparent simplicity, a complete analytical theory of Laplacian growth models is still lacking and most of the understanding comes from numerical studies.

[2] In ref. [27], an averaging of the square of the radius of gyration $r(t)$ as a function of time t has been performed over many walkers. In this case, the natural origin of time $t = 0$ acts as an effective "rephasing" scheme. This is similar to averaging over DLA cluster, as we perform below, by counting with increasing masses and not with increasing radius.

A problem that most theoretical approaches do not tackle is the existence of a microscopic cut-off, namely the size of the sticking particles. In other words, has DLA a continuum limit? Our results showing the presence of log-periodicity suggest that the answer is negative and that the microscopic scale cascades its way up the hierarchy. This concept has been explored in simplified Laplacian needle models [16], showing the existence of a cascade of ("ultra-violet") Mullins-Sekerka instabilities that produces an approximate period-doubling cascade leading to discrete scale invariance on average [16].

We revisit our previous work [15] using a new canonical averaging scheme. 350 large 10^6 particles off-lattice DLA clusters were analysed for log-periodic signatures by constructing the local fractal dimension as the logarithmic derivative of the mass $M(r)$ of the cluster *versus* the distance from the center r,

$$D_r \equiv \frac{d \ln M\,(r)}{d \ln r}. \tag{3}$$

Good and reliable estimates of derivatives of numerical and experimental data are in general very difficult to obtain [28] due to "noise" *etc.* An efficient way of reducing such problems is to parametrise the data locally by some function and then using the derivative of *that function* as an estimate of the derivative of the data. Since we are looking for signatures of log-periodic oscillations, *i.e.*, local extrema, we need a function which at least preserves the second moment of the data. A filter which does this is the Savitzky-Golay smoothening filter [28] used in [15] in order to obtain an estimate D_r for each cluster.

A representative example of the local fractal dimension D_r of a DLA cluster as a function of log-distance to the center of the cluster $\ln r$ is shown in figure 1. We clearly see that D_r exhibits noisy quasi-periodic oscillations as a function of $\ln r$. Furthermore, we see that the amplitude of the oscillations decreases as a function of the size of the cluster. In fact, plots of log-periodic oscillations in the local fractal dimension have previously been published for both the dielectric breakdown model and the off-lattice DLA model as well as on-lattice [24], surprisingly enough without any comments from the authors. In our previous analysis [15], each of the 350 clusters was analyzed individually in order to estimate the frequencies of the oscillations. The synthetis of the analysis was to record the frequency of the two best fits of an oscillating function for each cluster [15]. This is a rather consuming procedure in terms of computing power. In addition, the significance of the analysis for each cluster is in general not overwhelming and only the cumulative evidence from the 350 clusters led to an unambiguous conclusion. Hence, it would be nice to have an averaging scheme allowing a neater signature.

Upon averaging an ensemble of DLA clusters with the same radius, the oscillations disappear (see for instance figure 3 of Ref.[15]). We thus turn to a "canonical" averaging scheme. The "canonical averaging" of the 350 clusters has been performed as follows. First, we need to define a local reduced control parameter that plays the role of $\Delta = p - p_c^R$ before averaging. In DLA, the critical point corresponds to the

ratio $r/a \rightarrow +\infty$ where r is the observation scale and a is the individual particle size. For a given r, different cluster realizations will have different masses. These mass fluctuations correspond to the grand-canonical ensemble. We thus propose to carry out the canonical average by analyzing the local fractal dimension D as a function of the mass M, *i.e.* the averaging over the different clusters is to be performed for the same value of M. This transformation is a nonlinear mapping from r to $M(r)$.

In fact, this procedure is computationally a very natural choice since M is the natural counter for the number of particles that have stuck to the cluster. Roughly speaking, we thus examine $\langle r(M) \rangle$ versus M to find $1/D(M)$. The local fractal dimension D is thus averaged over all 350 clusters for the same value of the mass $M\,(r)$ by interpolating between adjacent points in the numerical data. Others have already used this averaging procedure (see for instance ref.[29]). In figure 2, we see that the amplitude of the log-perodic oscillations has been somewhat diminished but are still clearly visible.

The canonical averaging preserves the oscillations in contrast to the grand canonical scheme which destroys them completely. The result from a Lomb periodogram analysis of figure 2 is shown in figure 3. The Lomb periodogram allows us to extract on small series the main relevant frequencies with a remarkable accuracy (see [28]). The first peak is very clear and correspond to the lower frequency (≈ 0.6) obtained from the individual frequency analysis presented in [15]. The corresponding preferred scaling ratio λ is approximately 5. In ref. [15], this peak was interpreted as the square 2.3^2 of the fundamental scaling ratio found around 2.3. The second frequency found in [15] (≈ 1.3 corresponding to the fundamental scaling ratio $\lambda = 2.3$) carries some ambiguity since the peak has been split into two corresponding to a small frequency shift. Similar results have been found by averaging over different subsets of the ensemble of 350 clusters. In particular, the two main peaks of the Lomb periodogram are robust features.

This novel analysis strengthens the case for log-periodic oscillations in DLA clusters and more generally in Laplacian growth [16]. As it provides a relatively simple procedure to systematically analyze new data, we hope it will also stimulate other tests. Note that, in contrast, the available theories (apart from Ref.[16]) do not predict the existence of this discrete scale invariance. The reason is probably due to not taking into account the microscopic scale cut-off [30, 31].

The Yale group has presented a series of analysis [29, 32, 33] on fifty one-million particles and twenty ten-million particles clusters, that stress the importance of non-asymptotic effects and the existence of a substantial amount of sample fluctuations. While it seems delicate, without a testable theory, to extrapolate the observed decrease of the normalized moments and conclude that the data supports the “drift” scenario [32] of an infinitely continuing transient, we note that the logarithmic doubly coordinate figures presented in [29] exhibit significant systematic *oscillations*. In particular, figure 2 of ref.[29] of the local dimension as a function of the logarithm of the particle number, which is similar to our figures 1 and 2, exhibits very clearly four log-periodic oscillations with a (modulated) amplitude significantly larger than

the deviation from zero that the authors are trying to argue. Note that figure 2 of ref.[29] is constructed in the same manner as our figure 2 by averaging clusters with the same mass. We concur with ref.[33] that the mass-lacunarity effect has to be included, otherwise finite size analysis are bound to be seriously distorded. In the light of these and our analysis, it seems more probable that the most important source of lacunarity is reflected in a log-periodic behavior, which has a clear physical basis, relying on a cascade of Mullins-Sekerka period-doubling instabilities [16].

3 The thermal fuse model

Damage is another example of a growth process. Some quasi-static scalar models of rupture [34, 35] can in fact be put in exact correspondence with DLA. Here, we do not persue the analysis of the geometry of the clusters/cracks but turn to the time domain and study the time-dependence of precursory signals possibly announcing global failure. Time-to-failure analysis has a long history [37, 36] and is based on the detection of an acceleration of some measured signal, for instance acoustic emissions, on the approach to the global failure. For prediction purposes, the problem is that an acceleration for instance modelled by a power law in general provides a poor constraint on the actual time of global rupture. Log-periodic structures, however, have the potential to improve significantly the prediction ability since the parameters of the fitting function can "lock-in" on the oscillations to get a more constrained determination of the global rupture time. This was first tested on engineering fiber composite structures [18] for which log-periodic signatures on the acoustic energy radiated as a function of time-to-failure could be identified. Since then, this has been tested extensively on several tens of pressure tanks made of kevlar-matrix and carbon-matrix composites constructed by Aérospatiale Inc., France. The results from these tests indicate that a precision of a few percent in the determination of the stress at rupture is obtained using acoustic emission recorded 20 % below the stress at rupture. These results have warranted the selection of this non-destructive evaluation technique as the routine qualifying procedure in the industrial fabrication process.

Here, we would like to present a cleaner analysis that demonstrates the existence of log-periodicity in the time-to-failure signals. In this goal, we revisit the thermal fuse model [38] (initially introduced in an electric framework, this model is straightforwardly translated into its mechanical analog). This is a genuine dynamical model, with mode III antiplane elasticity, here put on a square lattice where the bonds are elastic elements with elastic compliances g_i uniformly distributed in the interval $[g - \Delta : g + \Delta]$. A force F is applied across two parallel boundaries of the lattice and periodic boundary conditions are assumed in the perpendicular direction. Due to the force, each bond is damaged progressively according a standard power law dependence. In addition, a degree of healing is permitted. The damage variable D_n

of bond n varies according to

$$\frac{dD_n}{dt} = g[f_n(t)]^b - aD_n(t)\,, \tag{4}$$

where the first term of the r.h.s. represent the damage rate due to the force f_n exerted on this bond and the second term is the healing term. The limit $b \to \infty$ recovers the quasi-static model of rupture where the most stressed bond ruptures [39]. In the simulations used here, $b = 2$, $a = 1$ and $g \in [1 \pm 0.9]$. At time 0, the force F is applied on the system. As a consequence, all bonds start to get progressively damaged. The bond whose damage variable reaches first 1 breaks down. The force that it supported is then immediately redistributed according to the law of (equilibrium) elasticity (long-range Green function), corresponding to a novel mechanical equilibrium (we thus deal with a so-called quasi-static model of rupture; the dynamics stems from the damage behavior.). As a consequence, all the forces in the remaining bonds are modified and their new values enter the damage rate equation (4). A second bond will break and so on.

An example of a simulation is shown in figure 4, where the last point corresponds to a complete disconnection or "total rupture". It has previously been established that the ensemble average of the total dissipated power, or equivalently of the released elastic energy E, behaves as a power law

$$E \sim (t_c - t)^{-\alpha}\,, \tag{5}$$

as a function of time to rupture $t_c - t$ [38]. This result was obtained through a (grand canonical) ensemble averaging over different realisations. It has also been confirmed experimentally on an analogous dynamical electric breakdown experimental composite system [40]. See figure 5 which shows the average over 19 simulations of the thermal fuse model of the elastic energy release rate or rate of broken bonds as a function of time. The average is here performed using the standard ("grand canonical") averaging scheme, in terms of the time t from the beginning of the rupture process.

However, when looking at individual data, we see fluctuations that suggest, similarly to the experimental pressure tank case, the existence of systematic log-periodic structures. In order to unravel them, we have re-analysed the data presented in [38] performing a canonical averaging procedure. Specifically, we identify the time t_c^r for *each* disorder realization, where the second derivative of E (representing a kind of susceptibility) had its maximum value. Then, the energy release rate is averaged over the 19 available realisations for the same value of $\Delta = t_c^r - t$. Due to the noisy nature of the data, see figure 4, the derivatives was calculated in the following fashion. First, the cumulative distribution of broken bonds or released energy as a function of time was calculated. Then the distribution was lightly smoothed with a moving average filter with a window of 11 points and the second derivative was calculated using the smoothen data. Before showing the result of the canonically averaged data, examine the ensemble average of the smoothened data in figure 5. As previously established

in [38], the average energy release follows a power law $E \sim (t_c - t)^{-\alpha}$ as the time of rupture T_c is approached. Hence, the smoothening of the data has not in any way added any new and artificial features. For comparison, we see in figure 6 the canonically averaged energy release rate as a function of $t_c^r - t$. Five approximately equidistant (in $\ln(t_c^r - t)$) peaks are visible. This means that the previous equation (5) should be replaced by

$$E \sim (t_c - t)^{-\alpha} P\left(\frac{\ln(t_c - t)}{\ln \lambda}\right), \tag{6}$$

where P is a periodic function of period 1. If we estimate the corresponding value of λ from the peaks shown in figure 6 we get $\lambda \approx 2.4 \pm 0.4$, *i.e.*, again close to 2 as in the case of DLA. That the DLA model and the thermal fuse model have approximately the same preferred scaling ratio is quite reassuring considering the close relationship between the two models.

3.1 Conclusion

We have studied two numerical models, the DLA model and the thermal fuse model. Instead of the usual ensemble averaging over different disorder realisations, a novel averaging scheme adapted to preserve subtle correlated fluctuations in the data have been used with remarkably success, particularly in the case of the rupture model.

The analysis presented here provides yet another example of spontaneous generation of discrete scale invariance in a growth model. This is quite remarkable considering that no pre-existing hierarchy exists in the formulation of the model. The discrete scale invariance embodied in (6) is created *dynamically*, *i.e.* self-organized through the interactions between the cracks.

As to the underlying physical origin of the observed log-periodic oscillations in DLA and the thermal fuse model, it is probably a cascade of crack tip instabilities as shown numerically in [16] with respect to simplified Laplacian growth models. We find it rather remarkable that log-periodicity is present in the random rupture model in particular in view of the fact that cracks nucleate all over the two-dimensional system in a disordered manner. However, the long-range nature of elastic interactions apparently provides a sufficiently strong ordering force.

Finally, our essential message in the light of the recent discussions [4, 11] and the results presented here is that it would be fruitful to revisit heterogeneous systems with special attention to the possible degradation of information induced by standard averaging procedures. It is our hope that this might stimulate the invention of novel approaches to the physics of random systems than that offered by the traditional tool of the field.

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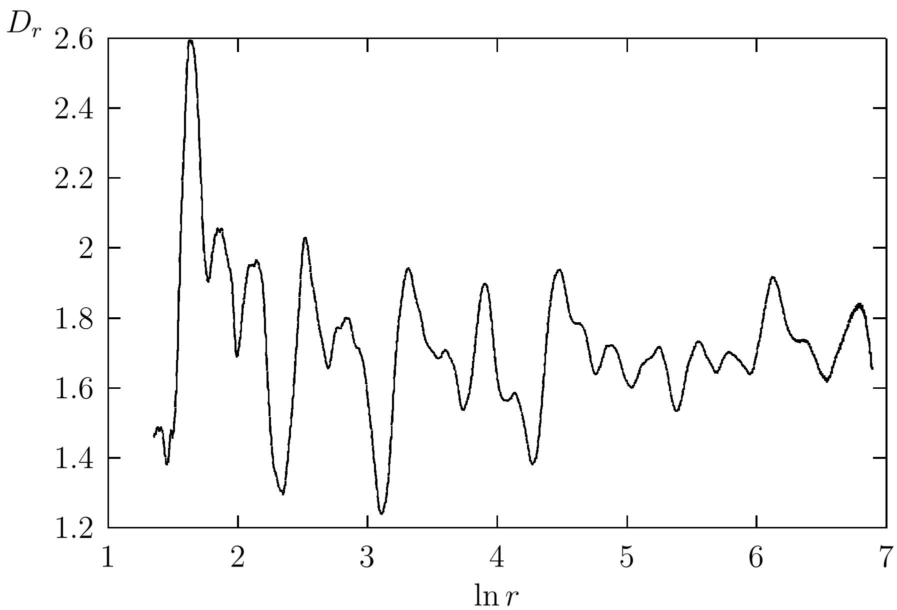

Figure 1: Example of the local dimension $D_r(\log r) = \frac{d \log M(r)}{d \log r}$ as a function of $\log r$ for a typical DLA cluster. The numerical estimate of the derivative has been obtained with a Savitsky-Golay smoothing filter.

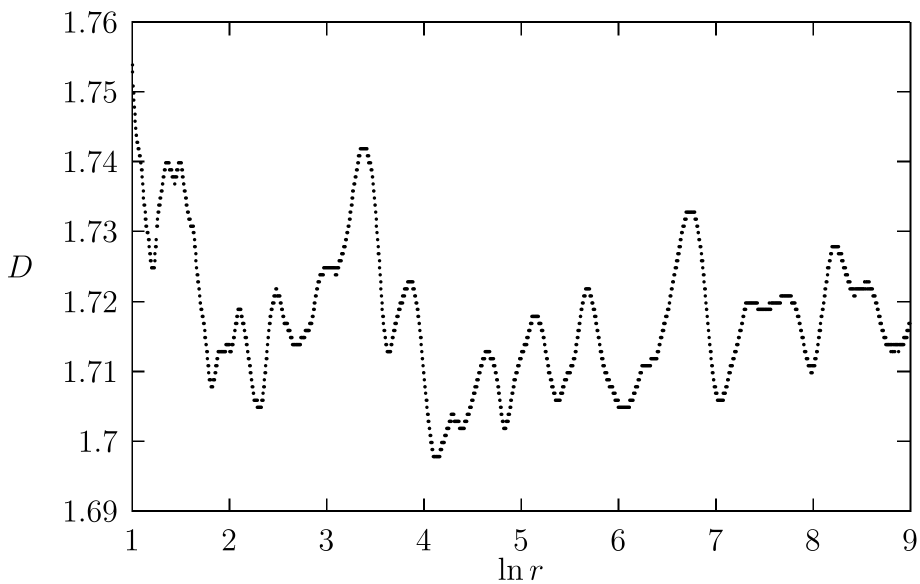

Figure 2: Canonical averaging of the local fractal dimension of 350 DLA clusters compared to the grand canonical averaging.

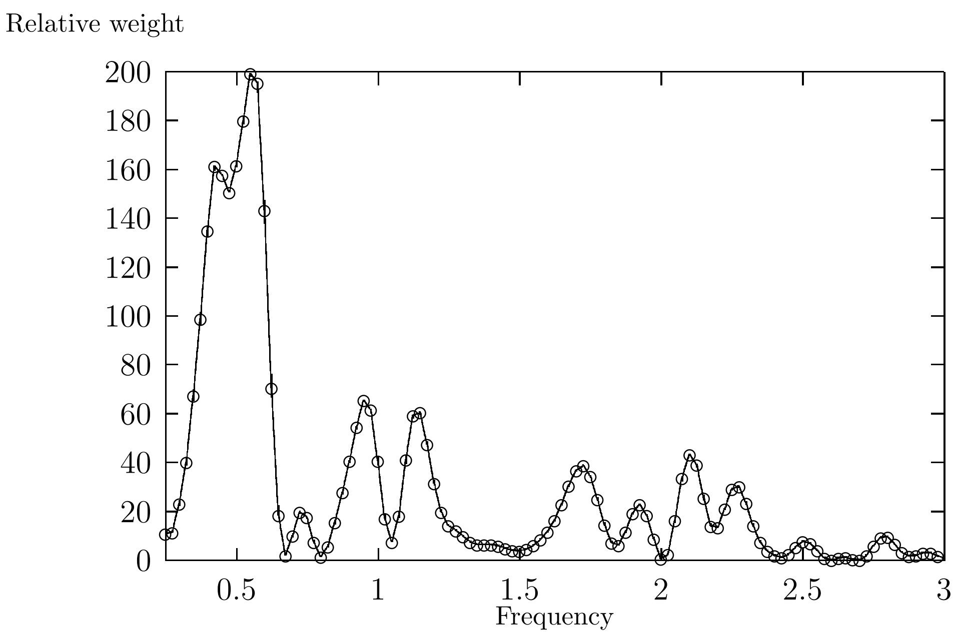

Figure 3: Lomb periodogram analysis of figure 2.

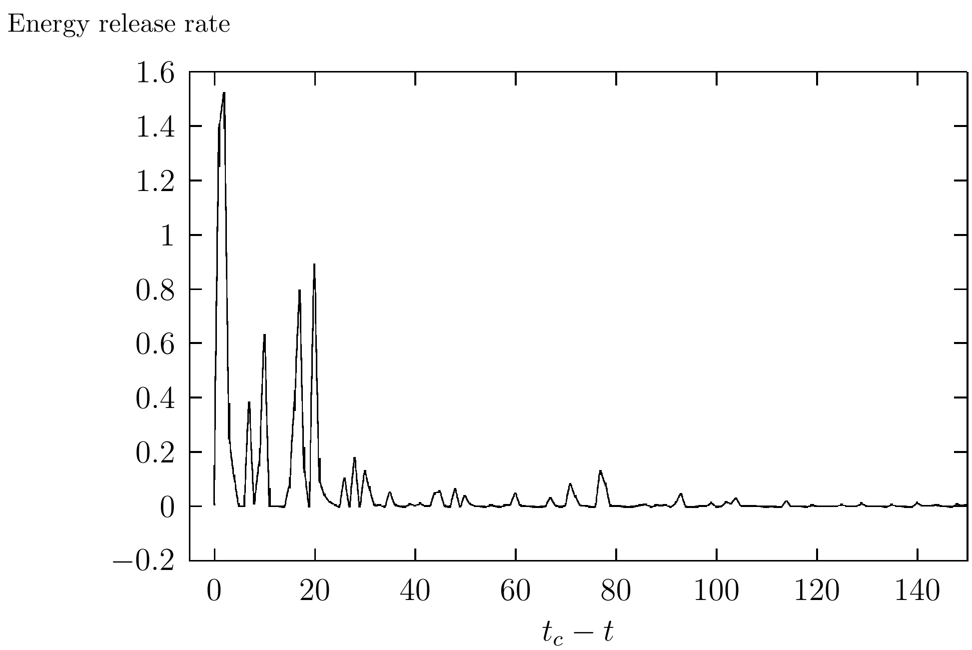

Figure 4: A single simulation of the thermal fuse model.

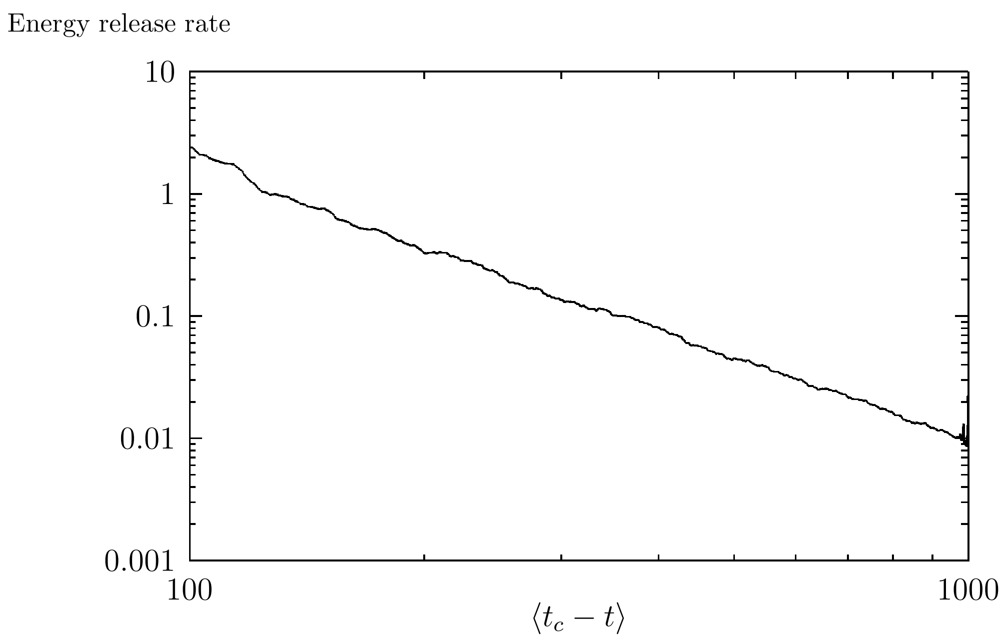

Figure 5: An average of the elastic energy release rate or rate of broken bonds as a function of time. The average is over 19 simulations of the thermal fuse model, using the standard (“grand canonical”) averaging scheme, in terms of the time t from the beginning of the rupture process.

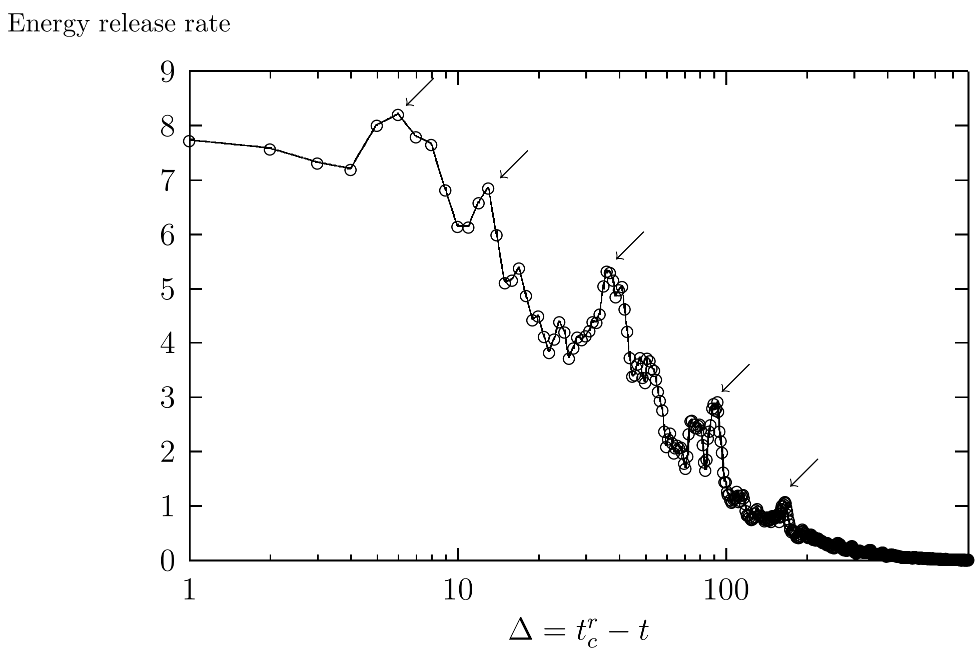

Figure 6: An average of the elastic energy release rate or rate of broken bonds as a function of time. The average is over the same 19 simulations of the thermal fuse model used in figure 5, but using the canonical averaging scheme described in the text, in terms of the effective time-to-failure measured by the sample specific susceptibility maximum.